Numerical Analysis and Computational Mathematics

Fall Semester 2024 – CSE Section

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Linear systems & Eigenproblems

Exercise I (MATLAB)

Consider the two dimensional truss bridge reported in Figure 1 composed by a triangular lattice of beams linked at some joints (nodes). We consider N_{nodes} nodes (an odd number). Each triangle is equilateral. We assume that all the beams possess the same elastic properties and that they can uniquely carry axial loads. External forces can be applied only at the nodes. Our goal is to compute the displacements of the bridge's nodes under the action of external forces. We index nodes using the integer i, with odd values of i denoting the bottom nodes, see Figure 1.

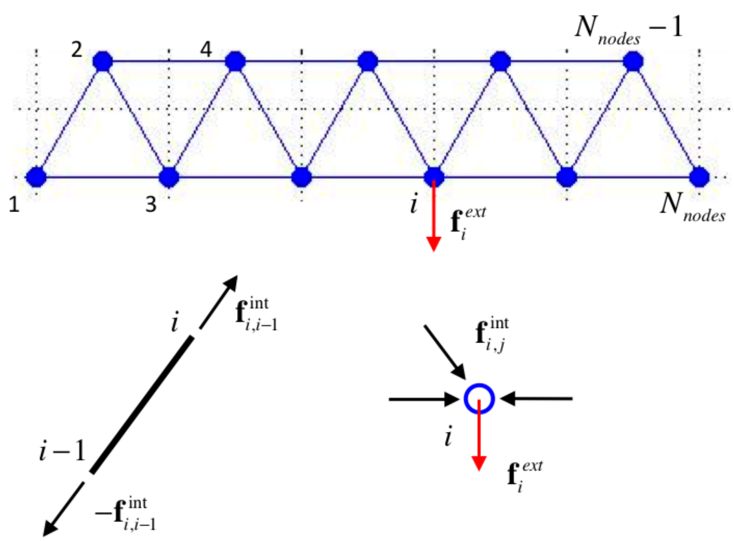

Figure 1: Truss bridge model.

We denote by $\mathbf{u}_i \in \mathbb{R}^2$ the displacement vector of each node $i = 1, \ldots, N_{nodes}$ and by $\mathbf{f}_i^{ext} \in \mathbb{R}^2$ the external force applied at node i. The (axial) internal forces $\mathbf{f}_{i,j}^{int}$ acting between the beams and the nodes depend on the displacements of the nodes i and j and the beams' orientations; they read:

$$\mathbf{f}_{i,i-1}^{int} = k_{beam} T_a (\mathbf{u}_i - \mathbf{u}_{i-1}) \qquad \text{for } i = 2, 4, 6, \ldots, N_{nodes} - 1,$$

$$\mathbf{f}^{int}_{i,i-1} = k_{beam} T_b (\mathbf{u}_i - \mathbf{u}_{i-1}) \qquad \text{for } i = 3, 5, 7, \ldots, N_{nodes},$$
$$\mathbf{f}^{int}_{i,i-2} = k_{beam} T_h (\mathbf{u}_i - \mathbf{u}_{i-2}) \qquad \text{for } i = 3, 4, 5, \ldots, N_{nodes},$$

where k_{beam} is the elastic constant of the beams and

$$T_a = \frac{1}{4}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & 3 \end{bmatrix}, \qquad T_b = \frac{1}{4}\begin{bmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & 3 \end{bmatrix}, \qquad T_h = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

To impose equilibrium of internal and external forces at each node of the bridge, we set

$$\sum_{j \in I_i} \mathbf{f}^{int}_{i,j} = \mathbf{f}^{ext}_i, \qquad I_i := \{i-2, i-1, i+1, i+2\} \cap \{k\}_{k=1}^{N_{nodes}}, \qquad \text{for all } i = 1, \ldots, N_{nodes}.$$

This leads to a discrete structural problem of the form

$$A\mathbf{d} = \mathbf{b},$$

where

$$\mathbf{d} = ((\mathbf{u}_1)^T, (\mathbf{u}_2)^T, \ldots, (\mathbf{u}_{N_{nodes}})^T)^T \in \mathbb{R}^{2N_{nodes}}$$

is the unknown solution vector representing the displacements of each node of the bridge. The matrix $A \in \mathbb{R}^{n \times n}$ is called the stiffness matrix and depends on the elastic properties of the beams and the pattern (the connectivity) of the lattice. The matrix A is sparse and has a block structure:

$$A = k_{beam} \begin{bmatrix} (T_h + T_a) & -T_a & -T_h & & & & \\ -T_a & (T_h + T_a + T_b) & -T_b & -T_h & & & \\ -T_h & -T_b & (2T_h + T_a + T_b) & -T_a & -T_h & & \\ & -T_h & -T_a & (2T_h + T_a + T_b) & -T_b & -T_h & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ & & & & & & \\ & & & & -T_h & -T_b & (T_h + T_b) \end{bmatrix}.$$

The vector

$$\mathbf{b} = ((\mathbf{f}^{ext}_1)^T, (\mathbf{f}^{ext}_2)^T, \ldots, (\mathbf{f}^{ext}_{N_{nodes}})^T)^T \in \mathbb{R}^{2N_{nodes}}$$

contains the external forces $\mathbf{f}^{ext}_i \in \mathbb{R}^2$ acting on each node $i = 1, \ldots, N_{nodes}$.

a) Set $N_{nodes} = 29$ and $k_{beam} = 10^3$. Build the stiffness matrix A in sparse format. Is A symmetric? Is it invertible? (*Hint*: use `condest`.) Note: you should verify your implementation by comparing your A with the matrix obtained via the provided `bridge_stiffness_matrix.m` function.

b) In order to make the structural problem well-posed we need to constrain at least three entries of the vector $\mathbf{d}$ by providing boundary conditions. We want to impose zero displacements of the leftmost node for both vertical and horizontal components, and a zero vertical displacement of the rightmost node. To this aim, extract a system $\widetilde{A}\widetilde{\mathbf{d}} = \widetilde{\mathbf{b}}$ of reduced size $2N_{nodes} - 3$, where the three above-mentioned entries of $\mathbf{d}$ have been removed. Is $\widetilde{A}$ invertible?

c) Impose downward external forces $\mathbf{f}^{ext}_i = (0, -1)^T$ at all the (odd-indexed) bottom nodes. Compute the corresponding displacement vector $\widetilde{\mathbf{d}}$. Visualize the deformed configuration of the bridge using the provided `plot_bridge.m` function. Note that `plot_bridge.m` requires $\mathbf{d}$ as input.

d) Assume that we are interested in determining the displacements $\mathbf{d}$ for several different external loads. Since the system matrix $\widetilde{A}$ stays the same while the right-hand-side $\widetilde{\mathbf{b}}$ changes, it makes sense to compute (only once) the LU factorization of $\widetilde{A}$ and then solve the triangular systems with different right-hand-sides. Do this for 10 different (random) external loads.

e) Repeat point c) using iterative methods to solve the linear system. First, use the MATLAB function `gmres` implementing the GMRES method. Then, use (without preconditioning) the MATLAB function `pcg` that implements the conjugate gradient method.

Exercise II (MATLAB)

Consider the matrix $A = \begin{bmatrix} 5 & -2 & -1 & 0 \\ -2 & 5 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ 0 & -1 & -1 & 5 \end{bmatrix}$.

a) Write a MATLAB function `power_method.m` that implements the power method to compute the largest (in magnitude) eigenvalue of a general matrix $A \in \mathbb{C}^{n \times n}$. Use the following template:

```
function [ lambda, x, k ] = power_method( A, x0, tol, kmax )
% POWER_METHOD power method for the computation of the largest eigenvalue
% (in modulus) of the matrix A (\lambda_1). We assume that A is square,
% |\lambda_1| > |lambda_i| for i=2,...n, and \lambda_1 non zero
% Stopping criterion based on the relative difference of successive
% iterates of the eigenvalue.
%  [ lambda, x ] = power_method( A, x0, tol, kmax )
%  Inputs: A      = matrix (n x n)
%          x0     = initial vector (n x 1)
%          tol    = tolerance for the stopping criterion
%          kmax   = maximum number of iterations
%  Output: lambda = computed (largest) eigenvalue
%          x      = computed eigenvector correspoding to lambda
%          k      = number of iterations
%

return
```

b) Compute the largest (in magnitude) eigenvalue of the matrix A using `power_method.m`, by setting $\mathbf{x}^{(0)} = (1, \dots, 1)^T$ and $tol = 10^{-6}$. Compare the obtained value with the "exact" one computed with `eig`.

c) Recall that the eigenvalues of A are the reciprocals of those of A^{-1} (if A is nonsingular). Use `power_method.m` to compute the smallest (in magnitude) eigenvalue of A.

d) Given a shift s, set $B = A - sI$, with I the identity matrix. Choose a value of s that turns the smallest (in magnitude) eigenvalue of A into the largest (shifted) eigenvalue of B. Then apply the power method to B to approximate the smallest (in magnitude) eigenvalue of A.

Exercise III (MATLAB)

Consider the structural bridge model from Exercise 1. Assume that each node of the bridge has mass m_{node}, while the beams are weightless. Define the diagonal matrix $M \in \mathbb{R}^{2N_{node} \times 2N_{node}} = \text{diag}(m_{node}, \dots, m_{node})$.

a) Set $N_{nodes} = 29$. Assemble the stiffness and mass matrices A and M, when $m_{node} = 1$ and $k_{beam} = 4 \cdot 10^2$. Then, constrain three displacements as in point b) of Exercise 1, obtaining reduced matrices of size $2N_{node}-3$. Consider the corresponding generalized eigenvalue problem $\widetilde{A}\widetilde{\mathbf{x}} = \widetilde{\lambda}\widetilde{M}\widetilde{\mathbf{x}}$. Verify that the matrices $\widetilde{A}$ and $\widetilde{M}$ are nonsingular.

b) Compute the 10 smallest (in magnitude) eigenvalues of the generalized eigenvalue problem using `eigs`. Using `plot_bridge.m`, visualize the corresponding eigenmodes.